

Relativistic mechanics of neutron superfluid in (magneto) elastic star crust.

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Abstract At densities below the neutron drip threshold, a purely elastic solid model (including, if necessary, a frozen-in magnetic field) can provide an adequate description of a neutron star crust, but at higher densities it will be necessary to allow for the penetration of the solid lattice by an independently moving current of superfluid neutrons. In order to do this, the previously available category of relativistic elasticity models is combined here with a separately developed category of relativistic superfluidity models in a unified treatment based on the use of an appropriate Lagrangian master function. As well as models of the purely variational kind, in which the vortices flow freely with the fluid, such a master function also provides a corresponding category of non-dissipative models in which the vortices are pinned to the solid structure.

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1 Introduction

The purpose of this work is to adapt the previously developed formalism for the treatment of various kinds of conductivity in a relativistic solid [1] to the particular case of a neutron superfluid in the solid crust of a neutron star. This treatment will include allowance for the presence of a frozen-in magnetic field in the limit of perfect electrical conductivity, which will be a very good approximation in this context. (Such an M.H.D. type limit is more relevant in neutron stars than the kind of diamagnetic polarisation that was considered in earlier work [2, 3].) The effects of magnetic fields may be significant for ordinary pulsars and will be particularly important in the special case of magnetars. In fact, the recently observed quasi periodic oscillations in the aftermath of giant flares in the soft gamma ray repeaters SGR 1806-20 [4, 5] and SGR 1900+14 [6] are suggested to be associated with oscillations originating in the crust after a major quake, see [7, 6, 8] and references therein. The magnetic field will be of crucial importance for a complete understanding of such oscillations.

One of the main motivations for this work is to deal with the frequency glitches in pulsars, whose treatment has long been recognised to require allowance both for elastic solidity and for the independent superfluid motion of neutrons in the crust at densities above the “drip” threshold (at about 10^{11}g/cm^3). In the literature on relativistic continuum mechanics, the superfluid aspects (whose development is relatively recent) have mainly been treated separately [9, 10, 11, 12, 13] from the purely elastic aspects, for which the formalism previously developed for that purpose [14, 15, 16, 17] has already been applied [18, 19, 20, 21, 22, 23, 24] to fairly realistic kinds of neutron star models (as well as to idealised configurations [25, 26] of more artificial kinds that are of essentially mathematical rather than astrophysical interest). A synthesis of the kind that is needed [27] for the treatment of neutron conduction through the solid crust has however been recently developed in a Newtonian framework [28], which should be accurate enough for the treatment at a local level of the outer and middle layers of a neutron star crust. Nevertheless, for the global treatment of a neutron star, and even for a local treatment of the deeper layers, a quantitatively precise description will require a fully relativistic treatment.

As a step toward the achievement of this purpose, the present work will develop a treatment that is general relativistic in the sense of being applicable to a space time background with an unrestricted Lorentzian metric with signature $\{-, +, +, +\}$ having components $g_{\mu\nu}$ with respect to arbitrary space-time coordinates x^μ . This metric will not be restricted by any specific evolution laws, so the treatment will be compatible not only with the standard Einstein theory of gravity but also with conceivable alternatives, and in particular with the trivial flat background case of special relativity, which will always be useful as an approximation in a local neighbourhood. We wish to stress that the formalism is not restricted to fixed background spacetimes. In particular, as an example, it is applicable to the full dynamical Einstein equations, for which it provides the right hand side.

The Newtonian treatment referred to above [28] included allowance for dissipative

effects such as vortex creep, but – to avoid introducing too many complications at once – the relativistic treatment provided here will be of conservative type, applying to the low temperature limit in which dissipative effects are neglected. As a generalisation of the variational formulation [15] of the purely elastic relativistic solid model [14] that is adequate for treating the outer crust layers (at densities below the “drip” threshold) the present treatment will be based on the use of an action density $\tilde{\Lambda}$ of a kind that can be used as a Lagrangian to obtain models of strictly variational type, but that can also be used (in the manner to described in Subsection 3.2) as a master function for the construction of non dissipative models of more general type such as are capable, as has already been shown in the relevant multiconstituent fluid limit case [10] of incorporating the potentially important effect of vortex pinning.

The category of conducting solid models developed here is designed for application on a macroscopic scale that is large compared with the intervortex separation. However as a special zero vorticity case, applicable on a mesoscopic scale – large compared with the microscopic scale of internuclear spacing, but small compared with the intervortex separation – this category includes models of the relatively simple hyperelastic subcategory [30] that is also of interest in a cosmological context.

The requisite Lagrangian master function $\tilde{\Lambda}$ will be formulated in terms of a solid structure, representing a distribution of atomic nuclei in terms of a 3-dimensional material base manifold with position coordinates q^A , together with a current n_f^μ of free baryons, representing superfluid neutrons, and a current n_c^μ of confined baryons, representing the protons and the fraction of the neutrons that are confined within the nuclei, so that their flow will be restricted by the comovement condition that is expressible (using a comma for partial derivation) by

$$n_c^\nu q_{,\nu}^A = 0. \quad (1)$$

2 Relativistic action formulation

2.1 Dynamic variables and background fields

As well as depending on the “live” dynamical fields, consisting in this case of the (vectorial) current components n_f^μ and n_c^μ and the (scalar) base space coordinates q^A and their derivatives, the Lagrangian will also depend on the metric $g_{\mu\nu}$ which in the present treatment will be considered as a “dead” – meaning predetermined – background. In such a case the most general infinitesimal Eulerian (i.e. fixed point) variation of the Lagrangian will be decomposable as the sum of two contributions in the form

$$\delta\tilde{\Lambda} = \delta^\heartsuit\tilde{\Lambda} + \delta^\ddagger\tilde{\Lambda} \quad (2)$$

in which $\delta^\heartsuit\tilde{\Lambda}$ is the realisable part attributable to a physically possible alteration of the configuration of the “live” dynamical fields, while $\delta^\ddagger\tilde{\Lambda}$ is a virtual part arising

from mathematically conceivable but (in the context under consideration) physically forbidden variations of the “dead” background fields that have been fixed in advance, of which the only one in the present instance is the flat or curved spacetime metric $g_{\mu\nu}$, so that the complete background variation contribution will be given just by

$$\delta^\dagger \tilde{\Lambda} = \frac{\partial \tilde{\Lambda}}{\partial g_{\mu\nu}} \delta g_{\mu\nu}. \quad (3)$$

The particular kind of background variation to be considered in this section is one that is simply generated by the action of a displacement field ξ^ν say, so that the ensuing field variations will be given just by the negatives of the corresponding Lie derivatives. For the background metric the resulting variation will be given (using round brackets indicate index symmetrisation) by an expression of the familiar form

$$\delta g_{\mu\nu} = -2\nabla_{(\mu} \xi_{\nu)}, \quad (4)$$

in which ∇_μ denotes the (components of the) Riemannian covariant differentiation operator with respect to $g_{\mu\nu}$.

In so far as the “live” contribution is concerned, a further subdivision arises in cases such as those of interest here, which are characterised by a constrained variation principle, meaning one whereby the “on shell” evolution condition is that of invariance of the relevant action integral with respect to a compactly supported perturbation of the dynamical fields that is not entirely arbitrary but constrained by an appropriate admissibility condition. In the applications under consideration, the relevant admissibility condition will be interpretable as the requirement for the change to represent a “natural” variation of the same given physical system, whereas more general changes would represent a replacement of the system by a slightly different system within the same category. In such a case the “live” field contribution will be decomposable (though not necessarily in a unique manner) as a sum of the form

$$\delta^\heartsuit \tilde{\Lambda} = \delta^\natural \tilde{\Lambda} + \delta^\sharp \tilde{\Lambda} \quad (5)$$

in which $\delta^\sharp \tilde{\Lambda}$ denotes a part that would be inadmissible for the purpose of application of the variation principle, whereas $\delta^\natural \tilde{\Lambda}$ is a “natural” variation that would be allowed for this purpose. This means that for any (unperturbed) configuration that is “on shell” – in the sense of satisfying the dynamical evolution equations provided by the constrained variation principle – a generic admissible variation must satisfy the condition

$$\delta^\natural \tilde{\Lambda} \cong 0, \quad (6)$$

using the symbol \cong to indicate equivalence modulo the addition of a divergence (which, by Green’s theorem, will give no contribution to the action integral from a variation that is compactly supported).

In the kind of medium with which we are concerned, the constrained variables will consist of the free and confined current vectors n_f^μ and n_c^μ , while the unconstrained fields

will consist of the triplet of scalar fields q^A that are interpretable as local coordinates on the material base manifold, and from which, together with their gradient components $q^A_{,\mu}$, an associated particle number (which might count ionic nuclei or crystalline solid lattice points in the relevant microscopic substructure) density current, n^ν_1 say, can be constructed in the manner prescribed below. The corresponding “live” part of the generic action variation will thus be expressible in the form

$$\delta^\heartsuit \tilde{\Lambda} = \mu^f_\mu \delta n^\mu_f + \mu^c_\mu \delta n^\mu_c + \mathcal{P}^\nu_A \delta q^A_{,\nu} + \frac{\partial \tilde{\Lambda}}{\partial q^A} \delta q^A, \quad (7)$$

in which μ^f_μ and μ^c_μ are interpretable as the 4-momentum covectors of the free and confined baryons respectively.

It is to be observed that the partial derivative $\mathcal{P}^\nu_A = \partial \tilde{\Lambda} / \partial q^A_{,\nu}$ has the noteworthy property of being tensorial not just with respect to transformations of the space time coordinates x^ν but also with respect to transformations $q^A \mapsto \tilde{q}^A$ of the material coordinates, q^A whose effect will simply be given by

$$q^A \mapsto \tilde{q}^A \quad \Rightarrow \quad \mathcal{P}^\nu_A \mapsto \mathcal{P}^\nu_B \frac{\partial q^B}{\partial \tilde{q}^A}, \quad (8)$$

This good behaviour contrasts with the comportment of the remaining partial derivative, $\partial \tilde{\Lambda} / \partial q^A$, for which the corresponding transformation has the non-tensorial form

$$\frac{\partial \tilde{\Lambda}}{\partial q^A} \mapsto \frac{\partial \tilde{\Lambda}}{\partial q^B} \frac{\partial q^B}{\partial \tilde{q}^A} + \mathcal{P}^\nu_C \tilde{q}^B_{,\nu} \frac{\partial^2 q^C}{\partial \tilde{q}^B \partial \tilde{q}^A}. \quad (9)$$

For each of the separate currents n^μ_f and n^μ_c the admissible variations are generated by flow world line displacements, which means that they will be specifiable by corresponding displacement vector fields, ξ^μ_f and ξ^μ_c say, as

$$\delta^\sharp n^\mu_f = -\mathcal{L}_{\xi_f} n^\mu_f - n^\mu_f \nabla_\nu \xi^\nu_f, \quad \delta^\sharp n^\mu_c = -\mathcal{L}_{\xi_c} n^\mu_c - n^\mu_c \nabla_\nu \xi^\nu_c, \quad (10)$$

in which the Lie derivative is just the commutator

$$\mathcal{L}_{\xi_f} n^\mu_f = \xi^\nu_f \nabla_\nu n^\mu_f - n^\nu_f \nabla_\nu \xi^\mu_f, \quad \mathcal{L}_{\xi_c} n^\mu_c = \xi^\nu_c \nabla_\nu n^\mu_c - n^\nu_c \nabla_\nu \xi^\mu_c. \quad (11)$$

For the material base space, the corresponding the flow world line displacements will be given just by unrestricted variations of the base coordinates q^A , which will be expressible in the form

$$\delta^\sharp q^A = -q^A_{,\mu} \xi^\mu, \quad (12)$$

in terms of their own displacement vector field, ξ^μ say, which can be chosen independently a priori, but which will subsequently be subjected to the identification $\xi^\mu = \xi^\mu_c$ when the confinement restriction (1) is imposed.

It can be seen (using integration by parts) that, in terms of the displacement fields introduced by (10) and (12), the generic admissible variation for the current carrying

medium will be expressible – modulo a variationally irrelevant divergence term – in the form

$$\delta^\sharp \Lambda \cong -f_\nu^f \xi_f^\nu - f_\nu^c \xi_c^\nu - f_\nu^S \xi^\nu, \quad (13)$$

in which the covectorial coefficient f_ν^f will be interpretable as the force density acting on the free neutron current, and f_ν^c will be interpretable as the force density acting on the confined baryon current, while the extra coefficient f_ν^S is the supplementary force density due, as discussed below, to stratification or solid elasticity, that acts on the underlying ionic (crystalline or glass like) lattice structure.

If we postulate that the system should obey the constrained variational principle that is expressible as the imposition of the “on shell” condition (6), then it is evident from (13) that the ensuing dynamical field equations will be expressible simply as the separate vanishing of each one of these force densities, i.e. as the requirement that we should have $f_\nu^f = 0$, $f_\nu^c = 0$ and $f_\nu^S = 0$. We shall however be mainly concerned with other possibilities, particularly cases in which one of the constituents, labelled “c” say, is “confined” in the sense of being subject to a convection condition of the form (1) which means that it has to move with the underlying ionic flow. When the application of the variation principle is subject to the corresponding convective constraint

$$\xi_c^\nu = \xi^\nu, \quad (14)$$

the ensuing system of dynamical field equations will no longer entail the separate vanishing of f_ν^S and f_ν^c , but only of the amalgamated ionic force density f_ν^I that is defined as their sum,

$$f_\nu^I = f_\nu^S + f_\nu^c, \quad (15)$$

which is interpretable as the net force density acting on the integrated system consisting of the ions in combination with the convected constituent.

2.2 Noether identity for general covariance

If we now replace the natural variations (10) by the corresponding Lie variations

$$\delta n_f^\nu = -\mathcal{L}_{\xi_f} n_f^\nu, \quad \delta n_c^\nu = -\mathcal{L}_{\xi_c} n_c^\nu, \quad \delta q^\Lambda = -\mathcal{L}_\xi q^\Lambda, \quad (16)$$

it can be seen that the current contributions will leave extra terms, so that we shall obtain a live variation of the form (5) with $\delta^\sharp \Lambda$ given by (13) and with the extra “unnatural” (variationally inadmissible) contribution given by

$$\delta^\sharp \tilde{\Lambda} = \mu_\nu^f \delta^\sharp n_f^\nu + \mu_\nu^c \delta^\sharp n_c^\nu, \quad \delta^\sharp n_f^\nu = n_f^\nu \nabla_\mu \xi_f^\mu, \quad \delta^\sharp n_c^\nu = n_c^\nu \nabla_\mu \xi_c^\mu. \quad (17)$$

We can now obtain a Trautman type Noether identity (that will hold regardless of whether or not the variational field equations are satisfied) by taking the variations of all the relevant fields, “live” as well as “dead”, to be given by the action of the same displacement field ξ^ν whose effect on the background was given by (4), so that

the resulting effect on the Lagrangian scalar will be given just by the corresponding displacement variation

$$\delta \tilde{\Lambda} = -\xi^\nu \nabla_\nu \tilde{\Lambda} \cong \tilde{\Lambda} \nabla_\nu \xi^\nu. \quad (18)$$

We thereby obtain a relation of the form

$$\xi^\nu (f_\nu^f + f_\nu^c + f_\nu^s) \cong \delta^\dagger \tilde{\Lambda} + \delta^\sharp \tilde{\Lambda} - \tilde{\Lambda} \nabla_\nu \xi^\nu. \quad (19)$$

in which, by (3), (4) and (17), the “dead” and “unnatural” parts will be respectively given by

$$\delta^\dagger \tilde{\Lambda} = -2 \frac{\partial \tilde{\Lambda}}{\partial g_{\mu\nu}} \nabla_\mu \xi_\nu, \quad \delta^\sharp \tilde{\Lambda} = (\mu_\nu^f n_f^\nu + \mu_\nu^c n_c^\nu) \nabla_\mu \xi^\mu. \quad (20)$$

The relation (19) can thereby be rewritten in the standard form

$$\xi^\nu (f_\nu^f + f_\nu^c + f_\nu^s) \cong -T^\mu{}_\nu \nabla_\mu \xi^\nu \cong \xi^\nu \nabla_\mu T^\mu{}_\nu, \quad (21)$$

in which the geometric stress momentum energy density tensor can be read out as

$$T^{\mu\nu} = 2 \frac{\partial \tilde{\Lambda}}{\partial g_{\mu\nu}} + \Psi g^{\mu\nu}, \quad \Psi = \tilde{\Lambda} - \mu_\nu^f n_f^\nu - \mu_\nu^c n_c^\nu. \quad (22)$$

Since the arbitrary field ξ^ν can be taken to be non zero only in the immediate neighbourhood of any chosen point, the Noether identity (21) implies that, at each point, this stress energy tensor will satisfy a divergence identity of the simple form

$$\nabla_\mu T^\mu{}_\nu = f_\nu^f + f_\nu^c + f_\nu^s. \quad (23)$$

In the “on shell” case for which the variational evolution equations are satisfied, the force density contributions on the right of (23) will all drop out.

2.3 Canonical formulation

If, instead of working out the variations in terms of force densities modulo a divergence as in (19) and (21), we insert the displacement contributions (20) and (16) into the “live” variation (7) directly in (2) we get an identity in which the coefficients of the locally adjustable fields ξ^μ and $\nabla_\mu \xi^\nu$ must vanish separately. The former of these conditions reduces to a triviality, but the latter provides a relation showing that the geometrically defined stress energy tensor (23) can be rewritten in the equivalent canonical form

$$T^\mu{}_\nu = \Psi \delta^\mu{}_\nu + \mu_\nu^f n_f^\mu + \mu_\nu^c n_c^\mu - \mathcal{P}^\mu{}_\nu, \quad (24)$$

in which the stress contribution at the end is given in terms of the coefficient introduced just before (8) by

$$\mathcal{P}^\mu{}_\nu = \mathcal{P}_A^\mu q_{,\nu}^A. \quad (25)$$

The “live” variation formula (7) also provides what is needed for the derivation of the corresponding expressions for the force densities introduced in (13) which for the fluid constituents will have the form that is already familiar from the preceding work [1]. In terms of the corresponding generalised vorticity 2-forms defined, using square brackets for index antisymmetrisation, by

$$\mathbf{w}_{\mu\nu}^f = 2\nabla_{[\mu}\boldsymbol{\mu}_{\nu]}^f, \quad \mathbf{w}_{\mu\nu}^c = 2\nabla_{[\mu}\boldsymbol{\mu}_{\nu]}^c, \quad (26)$$

these current force densities will be given by

$$\mathbf{f}_\nu^f = \mathbf{n}_f^\mu \mathbf{w}_{\mu\nu}^f + \boldsymbol{\mu}_\nu^f \nabla_\mu \mathbf{n}_f^\mu, \quad \mathbf{f}_\nu^c = \mathbf{n}_c^\mu \mathbf{w}_{\mu\nu}^c + \boldsymbol{\mu}_\nu^c \nabla_\mu \mathbf{n}_c^\mu, \quad (27)$$

while the force density acting on the underlying atomic structure of the medium will be given by the formula

$$\mathbf{f}_\mu^S = \frac{\delta\tilde{\Lambda}}{\delta q^A} q_{,\mu}^A, \quad (28)$$

in which the Eulerian derivative is given by a prescription of the usual form

$$\frac{\delta\tilde{\Lambda}}{\delta q^A} = \frac{\partial\tilde{\Lambda}}{\partial q^A} - \nabla_\nu \mathcal{P}_A^\nu, \quad \mathcal{P}_A^\nu = \frac{\partial\tilde{\Lambda}}{\partial q_{,\nu}^A}. \quad (29)$$

It is important to notice that the non-tensorial base coordinate transformation property (9) of the first term in this formula will be cancelled by that of the second, so that the effect on the Eulerian derivative of a change of the material base coordinates will be given simply by

$$q^A \mapsto \tilde{q}^A \quad \Rightarrow \quad \frac{\delta\tilde{\Lambda}}{\delta q^A} \mapsto \frac{\delta\tilde{\Lambda}}{\delta q^B} \frac{\partial q^B}{\partial \tilde{q}^A}, \quad (30)$$

which shows that the (stratification and solid elasticity) force density (28) is invariant with respect to such transformations, an important property that is not so obvious from its detailed expression

$$\mathbf{f}_\mu^S = \frac{\partial\tilde{\Lambda}}{\partial q^A} q_{,\mu}^A + \mathcal{P}_A^\nu \nabla_\mu q_{,\nu}^A - \nabla_\nu (\mathcal{P}_A^\nu q_{,\mu}^A), \quad (31)$$

in which only the final term is separately invariant.

It is important to observe that, regardless of what – variational or other – dynamical field equations may be imposed, the force density (31) will never be able to do any work on the medium since, as an obvious consequence of (28) it must automatically satisfy the further identity

$$\mathbf{f}_\nu^S \mathbf{u}^\nu = 0, \quad (32)$$

where \mathbf{u}^μ is the unit 4-velocity of the medium as specified by the defining conditions

$$q_{,\nu}^A \mathbf{u}^\nu = 0, \quad \mathbf{u}_\nu \mathbf{u}^\nu = -c^2, \quad (33)$$

where c is an adjustable parameter that is introduced to facilitate the derivation [29] of the Newtonian limit case.

If the variation principle (6) were imposed for the three independent displacement fields, so that – as remarked above – by (13) the three separate force densities f_μ^f , f_μ^c and f_μ^s would all have to vanish, then it evidently follows from the Noether identity (23) that the divergence condition

$$\nabla_\mu T^\mu{}_\nu = 0, \quad (34)$$

would have to hold. This condition is interpretable as an energy-momentum balance equation that must be satisfied whenever the system is not subject to any external forces. So long as we are concerned only with a system that is isolated in this sense, so that the energy-momentum balance condition (34) is satisfied, it follows from (23) that, even for a more general model admitting the presence of dissipative or other internal forces, these must be specified in such a way as to satisfy the total force balance condition

$$f_\nu^f + f_\nu^c + f_\nu^s = 0, \quad (35)$$

in order for the model to be self consistent. More particularly, in view of (32) it can be seen that the fluid force densities by themselves will have to satisfy the condition

$$u^\nu (f_\nu^f + f_\nu^c) = 0, \quad (36)$$

expressing energy balance in the local rest frame of the underlying medium.

3 Conservative dynamical conditions

3.1 Baryon conservation and the confinement constraint

The system under consideration is designed for applications in which the total baryon current

$$n_b^\nu = n_c^\nu + n_f^\nu, \quad (37)$$

can be decomposed as the combination of a free neutron current n_f^ν and a convected part n_c^ν consisting of the remaining baryons (some of the neutrons and all of the protons) that is convected with the nuclei, so that it is characterised by the confinement condition

$$n_c^\nu = n_c u^\nu, \quad n_c = (-n_{c\nu} n_c^\nu)^{1/2} / c. \quad (38)$$

As a consequence of the condition of conservation of the total baryon current,

$$\nabla_\nu n_b^\nu = 0, \quad (39)$$

it follows that in terms of the “free” current vorticity 2-form $w_{\mu\nu}^f$ the energy conservation identity (36) will reduce to the form

$$u^\mu w_{\mu\nu}^f n_f^\nu = u^\mu (\mu_\mu^f - \mu_\mu^c) \nabla_\nu n_f^\nu. \quad (40)$$

Due to the confinement constraint, there is a freedom of adjustment in the specification of the confined particle momentum covector μ_μ^c by the general formula (7), but this does not affect the quantities involved in (40) because there is no ambiguity in the specification of the energy component

$$\mathcal{E}^c = -u^\mu \mu_\mu^c. \quad (41)$$

An obvious way of resolving the ambiguity in the remaining (space) components of the confined particle momentum would be to set them to zero, so that the complete covector would simply take the form $\mu_\mu^c = -\mathcal{E}^c u_\mu$. Such a choice would be adequate for the specialised purpose matching the formalism used for the non-conducting solid limit case [29]. However for the broader objective of consistency with the formalism developed for the multiconstituent fluid limit case [9] we shall adopt a more satisfactory alternative ansatz for resolving the ambiguity in the specification of μ_μ^c , which is to fix it by imposition of a convention to the effect that

$$\mathcal{P}_A^\nu u_\nu = 0. \quad (42)$$

3.2 Vortex pinning and chemical equilibrium

As the condition of superfluidity of the “free” constituent will consist in a restriction of the possible behaviour of the corresponding momentum covector μ_μ^f , it is important to remark that this covector will not be affected by the chemical basis changes that will be discussed in Subsection 4.3. On a mesoscopic scale (large compared with the microscopic lattice spacing but small compared with intervortex separation) the condition of superfluidity will be expressed by the condition that this covector should have the form of a gradient, $\mu_\mu^f = \hbar \nabla_\mu \varphi$ of a periodic quantum phase angle φ , with the corollary that the corresponding “free” vorticity tensor $w_{\mu\nu}^f$ will also vanish at the mesoscopic level. At the macroscopic level, on a scale large compared with the separation between vortices, this vorticity will have a large scale average value that will not vanish, but to be compatible with a mesoscopic fibration by 2 dimensional vortex lines it must still be algebraically restricted by the degeneracy condition

$$w_{\mu[\nu}^f w_{\rho\sigma]}^f = 0. \quad (43)$$

This condition will automatically be satisfied by the field equations that will result from the relevant variation principle, which will require that the vanishing perturbation condition (6) should be satisfied, not for independent variations of all three of the displacement fields ξ_f^μ , ξ_c^μ , ξ^μ , which in view of the constraint (37) would lead to overdetermination, but just for variations satisfying the corresponding restraint (14). This requirement entails the vanishing, not of all three of the relevant force densities

$f_\mu^f, f_\mu^c, f_\mu^S$ given by the prescriptions (27) and (31), but only of the free force density f_μ^f and of the amalgamated force density f_μ^I given by (15) that acts on the part that is convected with the ionic lattice.

According to the general formula (27), the vanishing of the free force density f_μ^f entails not only the separate conservation of the free part n_f^μ of the current density but also a dynamic equation of the familiar form

$$w_{\mu\nu}^f n_f^\nu = 0, \quad (44)$$

which is interpretable as meaning that the vortex lines are “frozen” into the free fluid in the sense of being dragged along with it.

Instead of postulating the strict application of the variation principle, we can obtain a non-dissipative model of another physically useful kind by postulating that the vortices are “pinned” in the sense of being frozen, not into the fluid but into the atomic lattice structure, which means replacing (44) by a dynamical equation of the analogous form

$$w_{\mu\nu}^f u^\nu = 0, \quad (45)$$

which unlike (44) is unaffected by changes of the chemical basis [9] that is used, as discussed in Subsection 4.3, to specify which of the neutrons are to be considered to be “free”.

Whichever of the alternative possibilities (44) or (45) is adopted, the degeneracy condition (43) will evidently be satisfied, and more particularly the left hand side of the identity (40) will vanish, which means that the model must be such as to satisfy the condition

$$u^\mu (\mu_\mu^f - \mu_\mu^c) \nabla_\nu n_f^\nu = 0. \quad (46)$$

In the strictly variational case this requirement will obviously be implemented by the separate conservation condition

$$\nabla_\nu n_f^\nu = 0, \quad (47)$$

which will be physically realistic in the context of high frequency oscillations. However in applications to slow long term variations it will be more realistic to use a model based on the alternative possibility, namely that of chemical equilibrium, as given by the relation

$$u^\mu (\mu_\mu^f - \mu_\mu^c) = 0, \quad (48)$$

which, like (45) is does not depend on which fraction of the neutrons is considered to be “free”.

Thus, depending on how we choose between the alternatives (44) or (45), and between the alternatives (47) or (48), we can use the same Lagrangian master function $\tilde{\Lambda}$ for the specification of 4 different kinds of non-dissipative models, which are categorisable as unpinned with separate conservation or chemical equilibrium, and pinned with separate conservation or chemical equilibrium.

4 Internal action function

4.1 Subtraction of the ballistic contribution

For the purpose of studying the Newtonian limit it is convenient [29] to split up the total action function into parts depending just on the confined number density n_c defined by (38) and the corresponding free number density n_f defined by setting

$$n_f^\nu = n_f u_f^\nu, \quad n_f = (-n_{f\nu} n_f^\nu)^{1/2}/c, \quad (49)$$

in a decomposition of the form

$$\tilde{\Lambda} = \tilde{\Lambda}_{\text{bal}} + \Lambda_{\text{int}}, \quad (50)$$

(using the tilde just to distinguish the parts that will be relatively large in the non relativistic limit) in which the first term is a ballistic contribution (meaning that by itself it would simply imply geodesic motion of the separate contributions) given in terms of a fixed mass parameter m_o by an expression of the simple form

$$\tilde{\Lambda}_{\text{bal}} = -m_o c^2 (n_c + n_f), \quad (51)$$

so that all the non-trivial information about the system will be contained in the remaining internal action contribution Λ_{int} , which remains finite in the large c limit [29] (for which the tilde parts would diverge).

As far as the relativistic treatment is concerned, m_o may be given an arbitrarily chosen positive value or simply set to zero. The reason for taking the trouble of introducing it at all is that in appropriate circumstances m_o can be judiciously chosen in such a way as to make the internal remainder Λ_{int} relatively small, so that it can be dealt with as a small perturbation in an approximation of Newtonian type, for which m_o will be interpreted as the “rest mass per baryon”. In most astrophysical contexts for which a Newtonian approximation is applicable, and in particular for local perturbations in the crust of a neutron star, it will be good enough to use the proton mass calibration $m_o = m_p$, but it is well known that under laboratory conditions for which higher accuracy is needed it will be better to take the “rest mass per baryon” to be the standard atomic unit defined as a sixteenth of the mass of an ordinary O^{16} oxygen atom. At the opposite extreme, at extremely high densities it might seem more natural to interpret the “rest mass per baryon” as meaning three times the “bare mass” of a quark, but subtraction of the relatively small rest mass contribution defined in that way would be pointless. In practise, under the highly relativistic conditions prevailing in the core of a neutron star, the contribution from Λ_{int} will inevitably be relatively large, so that the most judicious choice may simply be to set $m_o = 0$.

However m_o is chosen, its introduction will lead to a corresponding momentum decomposition of the form

$$\mu_\nu^c = m_o u_\nu + \chi_\nu^c \quad \mu_\nu^f = m_o u_{f\nu} + \chi_\nu^f, \quad (52)$$

in which the non-trivial “chemical” momentum contributions from the internal part of the action can be read out from its expansion in the form

$$\delta^\heartsuit \Lambda_{\text{int}} = \chi^{\text{f}}_{\mu} \delta n^{\mu}_{\text{f}} + \chi^{\text{c}}_{\mu} \delta n^{\mu}_{\text{c}} + \mathcal{P}^{\nu}_{\text{A}} \delta q^{\text{A}}_{,\nu} + \frac{\partial \Lambda_{\text{int}}}{\partial q^{\text{A}}} \delta q^{\text{A}}, \quad (53)$$

in which (since the specification of the ballistic part does not involve the q^{A}) it can be checked that the coefficients in the last two terms will be the same as those already introduced in the analogous expansion (7) of the total action density.

4.2 Material projections

According to the general principle developed in the preceding work [1] the internal action will be expressible, at a given material position, as characterised by the material coordinates q^{A} , as a function just of the (time dependent) induced metric with contravariant components given by

$$\gamma^{\text{AB}} = g^{\mu\nu} q^{\text{A}}_{,\mu} q^{\text{B}}_{,\nu} = \gamma^{\mu\nu} q^{\text{A}}_{,\mu} q^{\text{B}}_{,\nu}, \quad \gamma^{\mu\nu} = g^{\mu\nu} + c^{-2} u^{\mu} u^{\nu}, \quad (54)$$

and of the relative current components

$$n^{\text{A}} = n^{\nu}_{\text{f}} q^{\text{A}}_{,\nu} = n^{\nu}_{\perp} q^{\text{A}}_{,\nu}, \quad n^{\nu}_{\perp} = \gamma^{\nu}_{\mu} n^{\mu}_{\text{f}} = n_{\parallel} [v]^{\nu}, \quad [v]^{\nu} u_{\nu} = 0, \quad (55)$$

together with the corresponding material rest frame number densities, namely the confined density n_{c} for which no projection is needed, and the free number density given in terms of the Lorentz factor γ specified by the relative current velocity $[v]^{\mu}$ as

$$n_{\parallel} = -c^{-2} u_{\nu} n^{\nu}_{\text{f}} = \gamma n_{\text{f}}, \quad \gamma = (1 - [v]^2/c^2)^{-1/2}. \quad (56)$$

These conditions imply that the internal action density (which will be the same as the complete action density if we choose $m_0 = 0$) will have a generic variation of the form

$$\delta \Lambda_{\text{int}} = -\chi^{\text{c}} \delta n_{\text{c}} - \chi^{\parallel} \delta n_{\parallel} + p^{\perp}_{\text{A}} \delta n^{\text{A}} + \frac{\partial \Lambda_{\text{int}}}{\partial \gamma^{\text{AB}}} \delta \gamma^{\text{AB}} + \frac{\partial \Lambda_{\text{int}}}{\partial q^{\text{A}}} \delta q^{\text{A}}. \quad (57)$$

This provides an associated “convective” variation [1] (in which, with respect to appropriately dragged coordinates, both q^{A} and $q^{\text{A}}_{,\nu}$ are held constant) of the form

$$\delta_{\text{c}} \Lambda_{\text{int}} = -\chi^{\text{c}} \delta_{\text{c}} n_{\text{c}} - \chi^{\parallel} \delta_{\text{c}} n_{\parallel} + \frac{\partial \Lambda_{\text{int}}}{\partial n^{\nu}_{\perp}} \delta_{\text{c}} n^{\nu}_{\perp} + \frac{\partial \Lambda_{\text{int}}}{\partial \gamma^{\mu\nu}} \delta_{\text{c}} \gamma^{\mu\nu}, \quad (58)$$

in terms of tensorial coefficients defined by

$$\frac{\partial \Lambda_{\text{int}}}{\partial n^{\nu}_{\perp}} = p^{\perp}_{\nu} = p^{\perp}_{\text{A}} q^{\text{A}}_{,\nu}, \quad \frac{\partial \Lambda_{\text{int}}}{\partial \gamma^{\mu\nu}} = \frac{\partial \Lambda_{\text{int}}}{\partial \gamma^{\text{AB}}} q^{\text{A}}_{,\nu} q^{\text{B}}_{,\mu}. \quad (59)$$

4.3 Ionic density and chemical gauge adjustments

It is computationally convenient and physically natural to take the material base space to be endowed with a measure form that is specifiable in terms of antisymmetric components, n_{IABC} say, that are fixed in the sense of depending only on the q^A , and that will determine a corresponding scalar space-time field n_{I} by the determinant formula

$$n_{\text{I}}^2 = \frac{1}{3!} n_{\text{IABC}} n_{\text{IDEF}} \gamma^{AD} \gamma^{BE} \gamma^{CF} . \quad (60)$$

For purposes of physical interpretation it will usually be convenient to take this measure to represent the number density of ionic nuclei, so that the confined baryon number density $n_{\text{c}} = -c^{-2} u_{\nu} n_{\text{c}}^{\nu}$ will be given by

$$n_{\text{c}} = A_{\text{c}} n_{\text{I}} , \quad (61)$$

where A_{c} is the atomic number, meaning the number of confined baryons (protons plus confined neutrons) per nucleus. It is to be observed that, whereas the number current n_{c}^{ν} will fail to be conserved in the chemical equilibrium case characterised by (48), the formalism is such that the corresponding ionic number current

$$n_{\text{I}}^{\nu} = n_{\text{I}} u^{\nu} \quad (62)$$

must automatically satisfy the conservation law

$$\nabla_{\nu} n_{\text{I}}^{\nu} = 0 \quad (63)$$

as a mathematical identity. Although there can be exceptions (for example when nuclear fission occurs) it will in fact most commonly be realistic to suppose that the number of ionic nuclei is indeed conserved even when the number of protons and neutrons they contain is changing (whether by transfusive interchange of the confined part or just by conservative “dripping” of neutrons into the ambient superfluid).

For particular physical applications, the specification of the part of the baryon current that is to be treated as “confined” will be to some extent dependent on the timescales of the processes under consideration as compared with the timescales for quantum tunnelling through the confining barriers containing the nuclei. For relatively rapid processes the fraction of the relevant quantum states that should be considered to be confined may be subject to an increase, so the corresponding current, \hat{n}_{c}^{ν} say, of baryons that are effectively confined will be somewhat larger than the value, n_{c}^{ν} , that would be appropriate for processes occurring over longer timescales. It is therefore of interest to consider the effect of such a chemical basis adjustment, as given by a transformation of the form

$$\hat{n}_{\text{c}}^{\nu} = (1 + \epsilon) n_{\text{c}}^{\nu} , \quad \hat{n}_{\text{f}}^{\nu} = n_{\text{f}}^{\nu} - \epsilon n_{\text{c}}^{\nu} , \quad (64)$$

for some suitably specified dimensionless adjustment parameter ϵ . So long as ϵ is just a constant, it is immediately apparent, that such a transformation will leave the “free” 4-momentum covector invariant, meaning that we shall have

$$\mu_{\nu}^{\text{f}} = \hat{\mu}_{\nu}^{\text{f}} , \quad (65)$$

and, as in the fluid case [9] it is also easy to see that the entire stress energy tensor(24) will be similarly invariant,

$$T^\mu{}_\nu = \hat{T}^\mu{}_\nu. \quad (66)$$

A less obvious observation [31] is that this covariance property of the canonical stress energy tensor is not restricted to transformations for which ϵ is constant. It will still hold whenever ϵ is specified as a function just of the confined atomic number $A_c = n_c/n_I$, and of the material position coordinates q^A . Since it can be seen from (60) that the variation of the ionic number will have a “live” (fixed background) part expressible in the form

$$\delta^\heartsuit n_I = n_I \gamma_{AB} q_{,\mu}^B \gamma^{\mu\nu} \delta q_{,\nu}^A + \frac{\partial n_I}{\partial q^A} \delta q^A, \quad (67)$$

the corresponding variation of the adjustment parameter will have the form

$$\delta^\heartsuit \epsilon = \frac{\partial \epsilon}{\partial A_c} \left(\frac{\delta^\heartsuit n_c}{n_I} - A_c \gamma_{AB} q_{,\mu}^B \gamma^{\mu\nu} \delta q_{,\nu}^A \right) + \left(\frac{\partial \epsilon}{\partial q^A} - \frac{\partial \epsilon}{\partial A_c} \frac{A_c}{n_I} \frac{\partial n_I}{\partial q^A} \right) \delta q^A. \quad (68)$$

Unlike the free particle momentum which is subject to the chemical invariance condition (65), the confined particle momentum covector will undergo a non-trivial chemical adjustment given by

$$\mu_\nu^c = \hat{\mu}_\nu^c + (\hat{\mu}_\mu^c - \hat{\mu}_\mu^f) \left(\epsilon \delta_\nu^\mu - \frac{A_c}{c^2} \frac{\partial \epsilon}{\partial A_c} u^\mu u_\nu \right). \quad (69)$$

However it can be verified that the extra terms will cancel out in the formula for the stress energy tensor so that the invariance condition (66) will remain valid.

4.4 Evaluation of momenta and stress energy tensor

To proceed from (57), what we need is the corresponding “live” variation, as carried out at a fixed position in a fixed background, for which one can obtain an expression of the standard form

$$\delta^\heartsuit \Lambda_{\text{int}} = \chi_\nu^c \delta n_c^\nu + \chi_\nu^f \delta n_f^\nu + \mathcal{P}_A^\nu \delta q_{,\nu}^A + \frac{\partial \Lambda_{\text{int}}}{\partial q^A} \delta q^A, \quad (70)$$

in which, by using the constraint $u^\nu \delta q_{,\nu}^A = -q_{,\nu}^A \delta u^\nu$ the terms can be recombined in such a way as to obtain a well defined set of coefficients characterised by the requirement that \mathcal{P}_A^ν should satisfy the condition (42). It can be seen that the required (purely spacelike) value of \mathcal{P}_A^ν will be given by

$$\mathcal{P}_A^\nu = p_A^\perp n_\perp^\nu + 2 \frac{\partial \Lambda_{\text{int}}}{\partial \gamma_{AB}} q_{,\mu}^B \gamma^{\mu\nu}, \quad (71)$$

while the corresponding expressions for the internal contributions to the free and confined and particle 4-momentum covectors will be given by

$$\chi_\nu^f = \frac{\chi^\parallel}{c^2} u_\nu + p_\nu^\perp, \quad \chi_\nu^c = \frac{\chi^c}{c^2} u_\nu - \frac{n_\parallel}{n_c} p_\nu^\perp + \frac{\chi^\parallel}{c^2 n_c} n_{\perp\nu}, \quad (72)$$

which implies that we shall have

$$\chi^\parallel = -u^\nu \chi_\nu^f, \quad \chi^c = -u^\nu \chi_\nu^c. \quad (73)$$

It follows, according to the canonical formula (24), that the internal contribution to the stress energy tensor will be given by

$$T_{\text{int}\nu}^\mu = \chi_\nu^f n_f^\mu + \chi_\nu^c n_c^\mu + \Psi \delta_\nu^\mu - \mathcal{P}_\nu^\mu, \quad (74)$$

with

$$\mathcal{P}_\nu^\mu = p_\nu^\perp n_\perp^\mu + 2 \frac{\partial \Lambda_{\text{int}}}{\partial \gamma^{\nu\rho}} \gamma^{\rho\mu}, \quad (75)$$

and with the generalised pressure scalar Ψ given, as in the multiconstituent fluid case, by

$$\Psi = \Lambda_{\text{int}} - \chi_\nu^f n_f^\nu - \chi_\nu^c n_c^\nu. \quad (76)$$

This can be rewritten in the manifestly symmetric standard form

$$T_{\text{int}}^{\mu\nu} = \rho_{\text{int}} u^\mu u^\nu + 2Q_{\text{int}}^{(\mu} u^{\nu)} + P_{\text{int}}^{\mu\nu}, \quad (77)$$

with

$$Q_{\text{int}}^\nu u_\nu = 0, \quad P_{\text{int}}^{\mu\nu} u_\nu = 0, \quad (78)$$

in which the symmetric internal pressure tensor and internal momentum density vector will be given by

$$P_{\text{int}}^{\mu\nu} = \Psi \gamma^{\mu\nu} - 2 \gamma^{\rho\mu} \gamma^{\sigma\nu} \frac{\partial \Lambda_{\text{int}}}{\partial \gamma^{\rho\sigma}}, \quad Q_{\text{int}}^\nu = \chi^\parallel n_\perp^\nu, \quad (79)$$

while finally the internal mass density ρ_{int} will be given by an expression of the pseudo-Hamiltonian form

$$c^2 \rho_{\text{int}} = p_\nu^\perp n_\perp^\nu - \Lambda_{\text{int}}. \quad (80)$$

To put the total into the same standard form

$$T^{\mu\nu} = \rho u^\mu u^\nu + 2Q^{(\mu} u^{\nu)} + P^{\mu\nu}, \quad (81)$$

whenever the rest mass parameter m_o is taken to be non-zero it will be necessary to use the correspondingly augmented quantities

$$P^{\mu\nu} = \frac{m_o}{n_f} n_\perp^\mu n_\perp^\nu + P_{\text{int}}^{\mu\nu}, \quad Q^\mu = m_o \gamma_\nu^\mu (n_c^\nu + n_c^\nu) + Q_{\text{int}}^\mu, \quad (82)$$

so that, finally, we have

$$\rho = m_o \left(n_c + \frac{n_{\parallel}^2}{n_f} \right) + \rho_{\text{int}} , \quad (83)$$

which can be rewritten as a Legendre type transformation formula giving the action density in terms of the rest mass energy density as

$$\tilde{\Lambda} = (m_o \gamma[v]_{\nu} + p_{\parallel \nu}) n_{\perp}^{\nu} - c^2 \rho . \quad (84)$$

5 Elastic energy contributions

5.1 Relaxed contribution

As in the multiconstituent fluid case [31] it is useful to decompose the internal action function in the form

$$\Lambda_{\text{int}} = \Lambda_{\text{ela}} + \Lambda_{\text{ent}} \quad (85)$$

so as to obtain a corresponding decomposition

$$\rho_{\text{int}} = \rho_{\text{ela}} + \rho_{\text{ent}} , \quad (86)$$

in which Λ_{ent} and ρ_{ent} are the parts attributable to entrainment (which will be dealt with in Section 6) and Λ_{ela} and ρ_{ela} are the static internal contributions that remain when the relative current contributions n^A are set to zero. For this static part (but not for the rest) the action density will just be the opposite of the elastic energy density ϵ as defined by setting

$$\epsilon = \rho_{\text{ela}} c^2 , \quad (87)$$

which can be seen from (80) to correspond to

$$\Lambda_{\text{ela}} = -\epsilon . \quad (88)$$

Whenever the comoving constituent can be considered to be a sufficiently good electrical conductor – as will be the case (on a macroscopic scale large compared with the scale of the relevant plasma frequency) in all the neutron star layers for which the present model is intended – it will be possible, in the manner recently described in the purely elastic case [29], to include allowance for electromagnetic coupling just in terms of a “frozen in” field, with antisymmetric material base-space components, F_{AB} say, that (like the components n_{IABC} of the ionic measure) will be fixed functions of the material position coordinates q^A . In such a case there will be a part, ϵ_{mag} say, of the elastic energy density that depends on the q^A not just directly but also via its dependence on the components F_{AB} , which themselves depend on the q^A , and it will

commonly be helpful to separate this magnetic part out for separate treatment (as described in Subsection 5.3) in a decomposition of the form

$$\mathcal{E} = \mathcal{E}_{\text{ins}} + \mathcal{E}_{\text{mag}} , \quad \Lambda_{\text{ela}} = \Lambda_{\text{ins}} + \Lambda_{\text{mag}} , \quad (99)$$

in which the \mathcal{E}_{ins} is the strictly static remainder (the only part that would be needed if the material were an electric insulator) that will be left when the frozen components F_{AB} are set to zero.

It will usually be convenient to further decompose the strictly static (meaning non magnetic) part of the elastic energy contribution in the form

$$\mathcal{E}_{\text{ins}} = \mathcal{E}_{\circ} + \mathcal{E}_{\text{sol}} , \quad \mathcal{E}_{\circ} = -\Lambda_{\circ} \quad (100)$$

in which \mathcal{E}_{\circ} is the part that remains in a relaxed configuration for which \mathcal{E} is minimised for given values of the independent current components n_{c} , n_{f} , n^{A} , and of the determinant of the induced metric γ^{AB} . Fixing this determinant is equivalent to fixing the value of the conserved number density n_{I} that is specified by (60). We shall use the notation $\check{\gamma}^{\text{AB}}$ and $\check{\gamma}_{\text{AB}}$ respectively for the corresponding relaxed values of γ^{AB} and its inverse γ_{AB} (as defined by $\gamma_{\text{AB}}\gamma^{\text{BC}} = \delta_{\text{A}}^{\text{C}}$) at which, for given values of n_{I} , n_{c} , n_{f} , the energy minimisation occurs. Thus substitution of $\check{\gamma}^{\text{AB}}$ for γ^{AB} in the solidity term \mathcal{E}_{sol} , or in the total \mathcal{E} , will give a generically reduced value

$$\check{\mathcal{E}}_{\text{sol}} \leq \mathcal{E}_{\text{sol}} , \quad \check{\mathcal{E}} \leq \mathcal{E} , \quad (101)$$

but it will have no effect on the relaxed part, or on the ionic number density n_{I} , for which we simply get

$$\check{\mathcal{E}}_{\circ} = \mathcal{E}_{\circ} , \quad \check{n}_{\text{I}} = n_{\text{I}} . \quad (102)$$

The relaxed contribution will evidently be of fluid type with a generic variation of the form

$$\delta\mathcal{E}_{\circ} = -\delta\Lambda_{\circ} = \chi_{\circ}^{\parallel} \delta n_{\parallel} + \chi_{\circ}^{\text{c}} \delta n_{\text{c}} + \chi_{\circ}^{\text{S}} \delta n_{\text{I}} - \lambda_{\circ\text{A}}^{\text{S}} \delta q^{\text{A}} , \quad (103)$$

(in which the final term allows for the possibility of built in inhomogeneity in addition to the stratification due just to the variation of the atomic number ratio A_{c}) with

$$\delta n_{\text{I}} = \frac{1}{2} n_{\text{I}} \gamma_{\text{AB}} \delta \gamma^{\text{AB}} + \frac{\partial n_{\text{I}}}{\partial q^{\text{A}}} \delta q_{\text{A}} . \quad (104)$$

The relaxed contribution to the pressure tensor as given by (79) will have an expression of the familiar isotropic form

$$P_{\circ}^{\text{AB}} = P_{\circ} \gamma^{\text{AB}} , \quad P_{\circ} = \chi_{\circ}^{\parallel} n_{\parallel} + \chi_{\circ}^{\text{I}} n_{\text{I}} - \mathcal{E}_{\circ} , \quad \chi_{\circ}^{\text{I}} = \chi_{\circ}^{\text{S}} + A_{\text{c}} \chi_{\circ}^{\text{c}} . \quad (105)$$

5.2 Solidity contribution

Let us now consider the solidity contribution \mathcal{E}_{sol} whose job is to allow for the effect (in \mathcal{E}_{ins}) of deviations of γ_{AB} from its relaxed value $\check{\gamma}_{AB}$ (as determined by the scalars n_f , n_c , n_I). Such deviations can conveniently be accounted for [14] in terms of the constant volume shear tensor whose material base-space representation is specified as

$$\varsigma_{AB} = \frac{1}{2}(\gamma_{AB} - \check{\gamma}_{AB}), \quad (96)$$

which means that the corresponding space time tensor will be given by

$$\varsigma_{\mu\nu} = \varsigma_{AB} q^A_{,\mu} q^B_{,\nu} = \frac{1}{2}(\gamma_{\mu\nu} - \check{\gamma}_{\mu\nu}). \quad (97)$$

In most applications to behaviour of a perfectly elastic (rather than plastic or other more complicated) kind, it will be sufficient to use an ansatz of quasi-Hookean type [14], meaning one in which the rigidity contribution has a homogeneously quadratic dependence on the deviation (96) in the sense that it will be given by an expression of the form

$$\mathcal{E}_{\text{sol}} = \frac{1}{2} \check{\Sigma}^{ABCD} \varsigma_{AB} \varsigma_{CD}, \quad (98)$$

with

$$\check{\Sigma}^{ABCD} = \check{\Sigma}_{(AB)(CD)} = \check{\Sigma}^{CDAB} \quad (99)$$

in which $\check{\Sigma}^{ABCD}$ is the relevant solidity tensor, for which the check symbol is used to indicate that, for a given value of the material position coordinates q^A , it depends only on the scalars n_f and n_c and (via n_I) on the relaxed metric $\check{\gamma}_{AB}$. The condition that γ_{AB} and $\check{\gamma}_{AB}$ must have the same determinant entails that on the 3 dimensional material base the symmetric shear tensor ς_{AB} will have only 5 (instead of 6) independent components, and more specifically that to first order it will be trace free with respect to either the actual metric γ_{AB} or the relaxed $\check{\gamma}_{AB}$. It is therefore necessary to impose a corresponding restriction to completely fix the specification of the solidity tensor $\check{\Sigma}^{ABCD}$, which can be conveniently done [14] by requiring that it be trace free with respect to the relaxed metric

$$\check{\Sigma}^{ABCD} \check{\gamma}_{CD} = 0. \quad (100)$$

It is to be remarked that an alternative procedure that would be equivalent for most practical purposes would be to work in an analogous manner with a shear tensor constructed as a difference between contravariant rather than covariant versions of the actual and relaxed forms of the induced metric. Another similarly equivalent possibility would be to use a judicious compromise between these alternatives, of the kind that has recently been shown [23] to be technically advantageous for certain purposes. In the small shear limit that is physically relevant, the differences between such alternatives are only of quadratic order $\mathcal{O}\{\varsigma^2\}$ for the shear itself and of higher order for the energy density \mathcal{E}_{sol} , and will therefore usually be too small to matter in practise, so the

question of whether to use a prescription of the originally proposed kind [14] or of a more sophisticated alternative kind [23] can be made just on the basis of mathematical convenience depending on the context of application.

The specification of the solidity tensor is not by itself sufficient to complete the specification of the elastic system, as it is also necessary to specify the dependence on n_I of the relaxed inverse metric $\check{\gamma}^{AB}$. The simplest possibility is that of what has been termed a perfect solid [14], meaning one for which the elastic structure at each material position is isotropic with respect to the relaxed metric, which in that case can vary only by a conformal factor. This means that it will be given in terms of its value γ_0^{AB} say at some fixed reference value n_0 say of the ionic number density n_I by

$$\check{\gamma}^{AB} = (n_I/n_0)^{2/3} \gamma_0^{AB}, \quad \check{\gamma}_{AB} = (n_0/n_I)^{2/3} \gamma_{0AB} \quad (101)$$

In the case of a solid structure that is isotropic (as will typically be the case on a macroscopic scale after averaging over randomly oriented mesoscopic crystals) the solidity tensor in the quasi Hookean ansatz will simply have to be given in terms of the relevant scalar shear modulus $\check{\Sigma}$ by the formula

$$\check{\Sigma}^{ABCD} = 2\check{\Sigma} (\check{\gamma}^{A(C} \check{\gamma}^{D)B} - \frac{1}{3} \check{\gamma}^{AB} \check{\gamma}^{CD}), \quad (102)$$

so that (98) will give the simple formula

$$\Lambda_{\text{sol}} = -\epsilon_{\text{sol}} = -\check{\Sigma} \varsigma^2, \quad (103)$$

in which the scalar shear magnitude ς is defined by the formula

$$\varsigma^2 = \check{\gamma}^{AB} \check{\gamma}^{CD} \varsigma_{BC} \varsigma_{DA} - \frac{1}{3} (\check{\gamma}^{AB} \varsigma_{AB})^2, \quad (104)$$

in which the final term will in practise be negligible since of quartic order, $\mathcal{O}\{\varsigma^4\}$, in the small ς limit that is relevant, as the trace is already of quadratic order, $\check{\gamma}^{AB} \varsigma_{AB} = \mathcal{O}\{\varsigma^2\}$. Under these conditions the solidity contribution will provide a pressure tensor given by the formula

$$P_{\text{sol}}^{AB} = -\check{\Sigma}^{ABCD} \varsigma_{CD} + P_{\text{sol}} \gamma^{AB}, \quad (105)$$

of which the final term is a pressure contribution given by

$$P_{\text{sol}} = \left(n_f \frac{\partial \check{\Sigma}}{\partial n_f} + n_c \frac{\partial \check{\Sigma}}{\partial n_c} + n_I \frac{\partial \check{\Sigma}}{\partial n_I} + \frac{\check{\Sigma}}{3} \right) \varsigma^2, \quad (106)$$

which, since it is of quadratic order in ς , will be negligible compared with the first (linear order) term for most practical purposes. (It is to be noted that the sign of the last of these quadratic order terms was one of the errata [17] in the original treatment [14].) The corresponding pressure adjustment contribution for the canonical formula (74) will be given by

$$\mathcal{P}_{\text{sol}}^\mu{}_\nu = \gamma_{\nu\lambda} \check{\Sigma}^{\lambda\mu\rho\sigma} \varsigma_{\rho\sigma} - \left(n_I \frac{\partial \check{\Sigma}}{\partial n_I} + \frac{4\check{\Sigma}}{3} \right) \varsigma^2 \gamma^\mu{}_\nu, \quad (107)$$

in which, again, the quadratic order term at the end will be negligible in practise.

5.3 Magnetic contribution

The presence of a frozen in magnetic field (whose effect is likely to be significant in applications to ordinary pulsars, and of paramount importance in the particular case of magnetars) is representable [29] by the specification of a fixed and closed material base-space two-form field with components F_{AB} . The closure condition means that it will be locally expressible in terms of a fixed gauge 1-form field with components A_A on the material base manifold as its exterior derivative

$$F_{AB} = 2A_{[B,A]} \quad (108)$$

The corresponding space-time tensor field

$$F_{\mu\nu} = F_{AB} q^A_{,\nu} q^B_{,\nu} \quad (109)$$

will therefore be given by

$$F_{\mu\nu} = A_{[\nu,\mu]}, \quad A_\nu = A_A q^A_{,\nu} \quad (110)$$

and will automatically satisfy the perfect conductivity condition

$$F_{\mu\nu} u^\nu = 0, \quad (111)$$

This condition is interpretable as meaning that field has no electric part, and so will be entirely determined by its magnetic part which will have material base components B^A given in terms of the alternating tensor ϵ^{ABC} associated with the induced metric γ^{AB} by

$$B^A = \frac{1}{2} \epsilon^{ABC} F_{BC}. \quad (112)$$

If polarisation effects can be neglected, which will probably be a good approximation in the crust (though not the core) of a neutron star, then the magnetic energy contribution will just be the negative of the usual vacuum action density associated with the field $F_{\mu\nu}$, which in terms of its materially projected components gives

$$\epsilon_{\text{mag}} = \frac{B^2}{8\pi}, \quad B^2 = \gamma_{AB} B^A B^B = \frac{1}{2} \gamma^{AB} \gamma^{CD} F_{AC} F_{BD}. \quad (113)$$

from which one obtains

$$\mathcal{P}_{\text{mag}}^{AB} = \frac{1}{4\pi} (B^A B^B - B^2 \gamma^{AB}). \quad (114)$$

It follows that the magnetic pressure tensor will be given by

$$P_{\text{mag}}^{AB} = \frac{1}{8\pi} (B^2 \gamma^{AB} - 2B^A B^B), \quad (115)$$

so that in terms of the magnetic field covector

$$B_\mu = \gamma_{AB} B^B q_{,\mu}^A, \quad (116)$$

one obtains an expression of the familiar form,

$$T_{\text{mag}}^{\mu\nu} = \frac{1}{8\pi} \left(\frac{B^2}{c^2} u^\mu u^\nu + B^2 \gamma^{\mu\nu} - 2B^\mu B^\nu \right), \quad (117)$$

for the magnetic stress energy tensor.

6 Effect of the current

6.1 Entrainment contribution

In general the entrainment action function Λ_{ent} will depend, for a given values of q^A , on the relative current components $n^A = q_{,\nu}^A n_\perp^\nu$ as well as on the scalar magnitudes n_f , n_c , and the induced metric components γ^{AB} so its generic variation will have the form

$$\delta\Lambda_{\text{ent}} = -\chi_{\text{ent}}^\parallel \delta n_\parallel - \chi_{\text{ent}}^c \delta n_c + \frac{\partial\Lambda_{\text{ent}}}{\partial\gamma^{AB}} \delta\gamma^{AB} + \frac{\partial\Lambda_{\text{ent}}}{\partial n^A} \delta n^A + \frac{\partial\Lambda_{\text{ent}}}{\partial q^A} \delta q^A. \quad (118)$$

The entrainment action Λ_{ent} is characterised by the condition that it vanishes when the relative current components n^A are set to zero, so when these components are sufficiently small, as will typically be the case, it will be a good approximation to take this contribution to have the homogeneous quadratic form

$$\Lambda_{\text{ent}} = \frac{1}{2n_\parallel} m_{AB}^\perp n^A n^B, \quad (119)$$

in which the entrainment mass tensor has components m_{AB}^\perp that, like the static action contribution, are independent of the current components n^A , so that the corresponding partial derivative in (118) will be given by

$$p_A^\perp = \frac{\partial\Lambda_{\text{ent}}}{\partial n^A} = \frac{1}{n_\parallel} m_{AB}^\perp n^B. \quad (120)$$

It is conceivable that the relaxed action function might involve a built in anisotropy favouring relative currents in some particular direction, but in cases of the simplest kind, to which the remainder of this subsection and the next will be restricted, this function Λ_{ent} will be of purely fluid type in the sense that for given values of q^A it will depend only on the set of four scalar magnitudes consisting of n_I , n_c , n_f together with the relative current magnitude n_\perp that is defined in terms of the (unrelaxed) metric

value γ_{AB} which – using the material index lowering operation specified by the induced metric γ^{AB} – will be given by

$$n_{\perp}^2 = n^A n_A, \quad n_A = \gamma_{AB} n^B, \quad \gamma_{AB} \gamma^{BC} = \delta_A^C. \quad (121)$$

Such a functional dependence provides an expansion

$$\delta\Lambda_{\text{ent}} = -\chi_{\text{ent}}^f \delta n_f - \chi_{\text{ent}}^c \delta n_c - \chi_{\text{ent}}^S \delta n_I + \lambda_{\text{ent}A}^S \delta q^A + \frac{\partial\Lambda_{\text{ent}}}{\partial n_{\perp}^2} \delta n_{\perp}^2, \quad (122)$$

of similar form to the perfect fluid contribution (93) but with an extra term involving a partial derivative that provides an expression of the form (120) in terms an isotropic mass tensor given by

$$m_{AB}^{\perp} = m_c^f \gamma_{AB}, \quad m_c^f = 2n_f \frac{\partial\Lambda_{\text{ent}}}{\partial n_{\perp}^2}, \quad (123)$$

while the other partial derivatives in (118) will be given by

$$\frac{\partial\Lambda_{\text{ent}}}{\partial\gamma_{AB}} = -\frac{1}{2} \left(\frac{m_c^f}{n_{\parallel}} n_A n_B + \chi_{\text{ent}}^S n_I \gamma_{AB} \right), \quad \frac{\partial\Lambda_{\text{ent}}}{\partial q^A} = \lambda_{\text{ent}A}^S - \chi_{\text{ent}}^S \frac{\partial n_I}{\partial q^A}. \quad (124)$$

The scalar m_c^f introduced in this way is identifiable as the increment $m_c^f = m_{\star} - m_o$ of the effective mass m_{\star} of the free baryons (meaning the superfluid neutrons) as compared with the ordinary baryonic mass m_o . This mass increment is expected to be positive (and in some layers large) [31] in the solid neutron star crust, but (moderately) negative in the fluid layers below.

6.2 Relaxed action contribution

It will be useful for many purposes to replace the decomposition (85) of the internal action density by an alternative decomposition of the form

$$\Lambda_{\text{int}} = \Lambda_{\text{lax}} + \Lambda_{\text{rig}}, \quad \Lambda_{\text{rig}} = \Lambda_{\text{sol}} + \Lambda_{\text{mag}}, \quad (125)$$

in which the relaxed – meaning shear independent – part will evidently consist of the combination

$$\Lambda_{\text{lax}} = \Lambda_{\circ} + \Lambda_{\text{ent}}. \quad (126)$$

The use of such a combination is particularly convenient whenever the entrainment contribution is of the isotropic type characterised by the variation expansion (122), in which case the complete relaxed action density will have a variation given by an expansion of the analogous form

$$\delta\Lambda_{\text{lax}} = -\chi_{\text{lax}}^f \delta n_f - \chi_{\text{lax}}^c \delta n_c - \chi_{\text{lax}}^S \delta n_I + \lambda_{\text{lax}A}^S \delta q^A + \frac{m_c^f}{2n_{\parallel}} \delta n_{\perp}^2, \quad (127)$$

with

$$\chi_{\text{lax}}^{\text{f}} = \chi_{\text{O}}^{\text{f}} + \chi_{\text{ent}}^{\text{f}}, \quad \chi_{\text{lax}}^{\text{c}} = \chi_{\text{O}}^{\text{c}} + \chi_{\text{ent}}^{\text{c}}, \quad \chi_{\text{lax}}^{\text{S}} = \chi_{\text{O}}^{\text{S}} + \chi_{\text{ent}}^{\text{S}}, \quad \lambda_{\text{lax A}}^{\text{S}} = \lambda_{\text{O A}}^{\text{S}} + \lambda_{\text{ent A}}^{\text{S}}. \quad (128)$$

It can be seen that the relaxed contribution to the “live” action variation (70) will simplify to provide an expression of the form

$$\delta^{\heartsuit} \Lambda_{\text{lax}} = \chi_{\text{lax} \nu}^{\text{f}} \delta n_{\text{f}}^{\nu} + \chi_{\text{lax} \nu}^{\text{c}} \delta n_{\text{c}}^{\nu} - \chi^{\text{S}} \delta^{\heartsuit} n_{\text{I}} + \lambda_{\text{lax A}}^{\text{S}} \delta q^{\text{A}}, \quad (129)$$

in which $\chi_{\text{lax} \nu}^{\text{f}}$ and $\chi_{\text{lax} \nu}^{\text{c}}$ can be read out as the relaxed parts of the internal momenta given by (72). Since we can write $\textcolor{violet}{c}^2 \delta^{\heartsuit} n_{\text{I}} = -u_{\nu} \delta^{\heartsuit} n_{\text{I}}^{\nu}$, it can be seen that, as the analogue of these free and confined particle 4-momentum contributions, we shall also be able to read out a corresponding ionic stratification 4-momentum contribution $\chi_{\text{lax} \nu}^{\text{S}}$, so as to obtain a complete set of relaxed internal momentum covectors that will be expressible in terms of the relative flow velocity vector introduced in (55) by

$$\chi_{\text{lax} \nu}^{\text{f}} = \frac{\chi_{\text{lax}}^{\parallel}}{c^2} u_{\nu} + \textcolor{red}{m}_{\text{c}}^{\text{f}} [v]_{\nu}, \quad (130)$$

$$\chi_{\text{lax} \nu}^{\text{c}} = \frac{\chi_{\text{lax}}^{\text{c}}}{\textcolor{violet}{c}^2} u_{\nu} + \frac{n_{\parallel}}{n_{\text{c}}} \left(\frac{\chi_{\text{lax}}^{\parallel}}{c^2} - \textcolor{red}{m}_{\text{c}}^{\text{f}} \right) [v]_{\nu}, \quad (131)$$

$$\chi_{\text{lax} \nu}^{\text{S}} = \frac{\chi_{\text{lax}}^{\text{S}}}{c^2} u_{\nu}, \quad (132)$$

It follows that the corresponding contribution to the generalised pressure scalar (76) will be given by

$$\Psi_{\text{lax}} = \Lambda_{\text{lax}} + n_{\parallel} \chi_{\text{lax}}^{\parallel} + n_{\text{c}} \chi_{\text{lax}}^{\text{c}} - \textcolor{red}{m}_{\text{c}}^{\text{f}} [v]^2. \quad (133)$$

In the formula (71) for the extra stress, it transpires that the contributions involving the mass increment $\textcolor{red}{m}_{\text{c}}^{\text{f}}$ (proportional to $\partial \Lambda_{\text{lax}} / \partial n_{\perp}^2$) will cancel out, leaving just the part due to stratification which will be given simply by

$$\mathcal{P}_{\text{lax} \mu}^{\nu} = -\chi_{\text{lax}}^{\text{S}} n_{\text{I}} \gamma_{\mu}^{\nu}. \quad (134)$$

It is to be observed that the need to include this extra term in the canonical stress energy formula can be avoided by treating the stratification momentum (132) as if it were on the same footing as the others, which means including a corresponding extra term in a supplemented pressure scalar Ψ^{S} defined by

$$\Psi^{\text{S}} = \Lambda_{\text{lax}} - \chi_{\text{lax} \nu}^{\text{f}} n_{\text{f}}^{\nu} - \chi_{\text{lax} \nu}^{\text{c}} n_{\text{c}}^{\nu} - \chi_{\text{lax} \nu}^{\text{S}} n_{\text{I}}^{\nu} = \Psi_{\text{lax}} + \chi_{\text{lax}}^{\text{S}} n_{\text{I}}. \quad (135)$$

Combining the relaxed internal contribution Λ_{lax} with the ballistic contribution $\tilde{\Lambda}_{\text{bal}}$ one obtains the complete liquid (meaning hydrodynamic as opposed to solid or magnetohydrodynamic) part of the action density in the form

$$\tilde{\Lambda}_{\text{liq}} = \tilde{\Lambda}_{\text{bal}} + \Lambda_{\text{lax}} = \tilde{\Lambda} - \Lambda_{\text{rig}}, \quad (136)$$

for which the corresponding liquid contributions to the momentum covectors (52) will be given (in terms of the relative flow velocity $[v]^\nu = n_\parallel^{-1} n_\perp^\nu$) by

$$\mu_{\text{liq}\nu}^{\text{f}} = m_{\text{o}} \gamma(u_\nu + [v]_\nu) + \chi_{\text{lax}\nu}^{\text{f}}, \quad \mu_{\text{liq}\nu}^{\text{c}} = m_{\text{o}} u_\nu + \chi_{\text{lax}\nu}^{\text{c}}, \quad \mu_\nu^{\text{S}} = \chi_{\text{lax}\nu}^{\text{S}}. \quad (137)$$

In terms of these quantities the corresponding liquid contribution to the stress energy tensor will be given simply by

$$T_{\text{liq}\nu}^\mu = \mu_{\text{liq}\nu}^{\text{f}} n_{\text{f}}^\mu + \mu_{\text{liq}\nu}^{\text{c}} n_{\text{c}}^\mu + \mu_{\text{liq}\nu}^{\text{S}} n_{\text{c}}^\mu + \Psi^{\text{S}} \delta_\nu^\mu, \quad (138)$$

and the corresponding elastically relaxed force contributions acting on free and confined currents will be given by an ansatz of the standard form (27) which gives

$$f_{\text{liq}\nu}^{\text{f}} = 2n_{\text{f}}^\mu \nabla_{[\mu} \mu_{\text{liq}\nu]}^{\text{f}} + \mu_{\text{liq}\nu}^{\text{f}} \nabla_\mu n_{\text{f}}^\mu, \quad (139)$$

and

$$f_{\text{liq}\nu}^{\text{c}} = 2n_{\text{c}}^\mu \nabla_{[\mu} \mu_{\text{liq}\nu]}^{\text{c}} + \mu_{\text{liq}\nu}^{\text{c}} \nabla_\mu n_{\text{c}}^\mu, \quad (140)$$

while for the analogously defined force contribution (28) due to stratification acting on the underlying ionic lattice, it can be seen that similar reasoning leads to an expression of the slightly different form

$$f_{\text{liq}\nu}^{\text{S}} = 2n_{\text{I}}^\mu \nabla_{[\mu} \mu_{\text{liq}\nu]}^{\text{S}} + \lambda_{\text{lax}\nu}^{\text{S}}, \quad \lambda_{\text{lax}\nu}^{\text{S}} = \lambda_{\text{lax}\Lambda}^{\text{S}} q_{,\nu}^{\Lambda}. \quad (141)$$

As remarked at the outset, because the confined particles and the ionic medium are constrained to move together the confined and supplementary force densities are not separately meaningful: all that is relevant for the equations of motion is their sum, the amalgamated ionic force density (15). Its liquid contribution can conveniently be expressed by introducing the corresponding amalgamated ionic 4-momentum that is definable as

$$\mu_{\text{liq}\nu}^{\text{I}} = \mu_{\text{liq}\nu}^{\text{S}} + A_{\text{c}} \mu_{\text{liq}\nu}^{\text{c}}, \quad (142)$$

(where $A_{\text{c}} = n_{\text{c}}/n_{\text{I}}$ is the atomic number as introduced above). In terms of the amalgamated ionic 4-momentum it can be seen that one will simply obtain the formula

$$f_{\text{liq}\nu}^{\text{I}} = 2n_{\text{I}}^\mu \nabla_{[\mu} \mu_{\text{liq}\nu]}^{\text{I}} + n_{\text{I}}^\mu \mu_{\text{liq}\nu}^{\text{c}} \nabla_\mu A_{\text{c}} + \lambda_{\text{lax}\nu}^{\text{S}}, \quad (143)$$

while the associated contribution to the stress energy tensor (24) will be expressible by the neatly concise formula

$$T_{\text{liq}\nu}^\mu = n_{\text{f}}^\mu \mu_{\text{liq}\nu}^{\text{f}} + n_{\text{I}}^\mu \mu_{\text{liq}\nu}^{\text{I}} + \Psi^{\text{S}} \delta_\nu^\mu, \quad (144)$$

in which we shall have

$$\Psi^{\text{S}} = \Lambda_{\text{lax}} - n_{\text{f}}^\rho \pi_{\text{lax}\rho}^{\text{f}} - n_{\text{I}}^\rho \pi_{\text{lax}\rho}^{\text{I}}. \quad (145)$$

6.3 Complete description for perfect conducting solid.

Let us conclude by putting all the pieces together for the case of what we refer to as a perfect conducting solid – meaning a model of the qualitatively simplest kind whose structure is fully isotropic with respect to the relaxed metric. In so far as the liquid part is concerned, this condition means not only that the incremental mass matrix has to be isotropic, as supposed in the preceding subsection, but also that the extra stratification vector in (141) must vanish, $\lambda_{\text{ax}\nu}^{\text{S}} = 0$.

Replacing the relaxed contribution (142) in (144) by the corresponding total ionic 4-momentum covector given (in terms of the confined atomic number A_c) by

$$\mu_{\nu}^{\text{I}} = \mu_{\nu}^{\text{S}} + A_c \mu_{\nu}^{\text{c}}, \quad (146)$$

we can apply an analogous tidying up operation to the complete stress energy tensor (24) which will thereby acquire the form

$$T_{\nu}^{\mu} = (\Lambda - n_{\text{f}}^{\rho} \mu_{\rho}^{\text{f}} - n_{\text{I}}^{\rho} \mu_{\rho}^{\text{I}}) \delta_{\nu}^{\mu} + n_{\text{f}}^{\mu} \mu_{\nu}^{\text{f}} + n_{\text{I}}^{\mu} \mu_{\nu}^{\text{I}} + T_{\text{mag}\nu}^{\mu} - \mathcal{P}_{\text{sol}\nu}^{\mu}, \quad (147)$$

in which the magnetic contribution is given by (117) and the only manifest allowance for the effects of solid rigidity is in the final term.

If we make the further realistic (and chemically invariant) supposition that (since it arises just from Coulomb interactions between the confined protons) the shear modulus depends only on n_c and n_{I} , then the solidity term will not contribute to the free particle momentum so we shall have

$$\mu_{\nu}^{\text{f}} = \mu_{\text{liq}\nu}^{\text{f}}, \quad f_{\nu}^{\text{f}} = f_{\text{liq}\nu}^{\text{f}}, \quad (148)$$

while the force density acting on the part that is comoving with the ions will be given by

$$f_{\nu}^{\text{I}} = f_{\text{liq}\nu}^{\text{I}} + f_{\text{rig}\nu}^{\text{I}}, \quad (149)$$

where the rigidity term (allowing for the combined effect of ionic solidity and the magnetic field that may be frozen into the accompanying degenerate electron plasma) will be given by

$$f_{\text{rig}\nu}^{\text{I}} = \nabla_{\mu} T_{\text{rig}\nu}^{\mu}, \quad T_{\text{rig}\nu}^{\mu} = T_{\text{mag}\nu}^{\mu} + T_{\text{sol}\nu}^{\mu}. \quad (150)$$

To linear order in the shear amplitude – which will be sufficiently accurate for most practical purposes – the solidity contribution will be given by the expression

$$T_{\text{sol}\nu}^{\mu} = -\mathcal{P}_{\text{sol}\nu}^{\mu} + \mathcal{O}\{\varsigma^2\}, \quad (151)$$

with $\mathcal{P}_{\text{sol}\nu}^{\mu}$ given by the perfect solid formula (107), which will itself be expressible to first order in the shear amplitude ς by a prescription of the familiar form

$$\mathcal{P}_{\text{sol}\nu}^{\mu} = 2\tilde{\Sigma} \varsigma_{\nu}^{\mu} + \mathcal{O}\{\varsigma^2\}, \quad (152)$$

in which $\tilde{\Sigma}$ is the ordinary shear modulus (which elsewhere is commonly denoted by the symbol μ that, in the present context, has already been used for the designation

of the material momentum components). This means that for practical purposes, for a conducting solid of the perfect (meaning intrinsically isotropic) type the term that is needed [27] to allow for deviation from behaviour of (multiconstituent) fluid type will be entirely contained in the magnetic contribution of the usual magnetohydrodynamic form given by (117) and the extra solidity term as given by (152), on the understanding that $\tilde{\Sigma}$ is prescribed as a function just of n_c and n_l .

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